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ON CONVOLUTION POWERS ON SEMIDIRECT PRODUCTS

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ABSTRACT

Let K be a compact group of linear operators of the d-dimensional space $\mathbf{R}^d$ and $G_{K,d}$ denote the semidirect product$K$ by $\mathbf{R}^d$. It is shown that if an adapted probability measure μ on $G_{K,d}$ is not scattered (i.e. for some compact F we have $\lim_{n\to\infty} \sup_{g\in G_{K,d}} \mu^{\star n}\ (gF) > 0$), then there exists a nonzero vector $x_0 \in \mathbf{R}^d$ such that $k_1(x_0) = k_2(x_0)$ holds for all (k_1, x_1) and (k_2, x_2) belonging to the topological support $S(\mu)$ of the measure μ. As a result we obtain that every adapted and strictly aperiodic probability measure on the group of all rigid motions of the d-dimensional Euclidian space is scattered.

Let G be a locally compact Hausdorff group and $P(G)$ denote the set of all (Radon) probability measures on G. For a fixed measure $\mu \in P(G)$ the question under what conditions on μ

$$(\star)\qquad \lim_{n\to\infty} \sup_{g\in G} \mu^{\star n}(gF) = 0 \qquad (F \text{ compact})$$

holds was studied by K.H. Hofmann and A. Mukherjea in 1981. They conjectured (see [HM]) that $(\star)$ is satisfied for all μ such that the smallest closed semigroup generated by the topological support $S(\mu)$ of μ is the whole group G (irreducibility). Recently a positive answer to this question has been provided in [W] by G. Willis.

A lasting challenge is to prove $(\star)$ under weaker assumptions than irreducibility. Let us recall that μ is said to be **adapted** if the smallest closed subgroup $G(\mu)$

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containing $S(\mu)$ is the whole group G. By $h(\mu)$ is usually denoted the smallest closed subgroup H of G such that for all $g \in S(\mu)$ we have $\mu(gH) = 1$ and $gH = Hg$. If μ is adapted then $h(\mu)$ has been identified in [DL] as the closed subgroup generated by $\bigcup_{n=1}^{\infty} S(\check{\mu}^{\star n} \star \mu^{\star n}) \cup S(\mu^{\star n} \star \check{\mu}^{\star n})$, where $\check{\mu}$ denotes the symmetric reflection of μ. In the sequel we shall deal with another subgroup related to μ. Namely, we define

$$G_{\mathrm{sym}}(\mu) = \overline{\bigcup_{j=1}^{\infty} \bigcup_{\varepsilon_k = \pm 1,\ \sum_{k=1}^{2j} \varepsilon_k = 0} S(\mu)^{\varepsilon_1} \cdot S(\mu)^{\varepsilon_2} \cdots S(\mu)^{\varepsilon_{2j}}}.$$

It is not difficult to notice that always $h(\mu) \subseteq G_{\mathrm{sym}}(\mu) \subseteq G(\mu)$. If $h(\mu) = G$ we say that μ is **strictly aperiodic**. In [DL] there is a list of groups G for which $(\star)$ holds for any adapted and strictly aperiodic probability measure μ. For instance, all nonamenable and nonunimodular groups are there. By recent results of the author (see [B]) groups which possess invariant metrics may be added.

In the end of the paper [DL] a suggestion is formulated that perhaps $(\star)$ is not fulfilled for some strictly aperiodic and adapted measure $u \in P(G)$, where G is the group of all rigid motions of the Euclidian space $\mathbf{R}^3$. The aim of this note is to provide an answer (see Corollary 2) to this question. Namely, it again appears that $(\star)$ holds if μ is strictly aperiodic and adapted. Let us emphasize that our result is somewhat stronger, as instead of the group of rigid motions, we consider semidirect products of compact groups by $\mathbf{R}^d$. Before we will deal with these special groups we formulate three auxiliary lemmas which are valid for general groups. The first two of them are known and may be found respectively in [DL] and [B].

LEMMA 1: *Let μ be a probability measure on G. Then the following conditions are equivalent:*

(i) *$\mu^{\star n} \to 0$ vaguely (i.e. $\lim_{n\to\infty} \mu^{\star n}(F) = 0$ for any compact F),*
(ii) *$G(\mu)$ is noncompact.*

Following [B] we say that $\mu \in P(G)$ is **scattered** if $(\star)$ holds. On the other hand, if there exist a sequence $g_n \in G$ and a compact set $F \subseteq G$ such that $\mu^{\star n}(g_n F) = 1$ for all $n \in \mathbf{N}$, then μ is called **concentrated**. If both the two measures μ and $\check{\mu}$ are scattered (concentrated), then we say that μ is **s.scattered** (**s.concentrated**).

LEMMA 2 (see also [E]): *If μ is not scattered then there exists a probability measure $\rho \in P(G)$ such that $\check{\mu}^{\star n} \star \mu^{\star n} \Rightarrow \rho$ in the weak measure topology and*

$$\check{\mu} \star \rho \star \mu = \rho. \tag{1}$$

To prove the main result of the paper we will need the following:

LEMMA 3: *Let G be a compact metric group. Then for every $\mu \in P(G)$ the limit $\rho = \lim_{n\to\infty} \check{\mu}^{\star n} \star \mu^{\star n}$ is the Haar measure on $h(\mu)$ and moreover*

$$h(\mu) = \overline{\bigcup_{n=1}^{\infty} S(\check{\mu}^{\star n} \star \mu^{\star n})} = S(\rho) = G_{\mathrm{sym}}(\mu).$$

Proof: By P_μ we denote the convolution operator $P_\mu f(\,\cdot\,) = \int f(\,\cdot\, t) d\mu(t)$. We notice that P_μ is a positive linear contraction on $L^2(G)$. By compactness of G the iterates $\{P_\mu^n f\}_{n\geq 1}$ of any fixed $f \in L^2(G)$ are relatively compact for the norm $\|\cdot\|_2$ of $L^2(G)$. It implies that the set

$$E_f = \{\, t \in G\colon \|T(t)P_\mu^n f - P_\mu^n f\|_2 \to 0\}$$

is closed in G, where $T(t)$ stands for the isometry $T(t)\tilde{f}(\,\cdot\,) = \tilde{f}(\,\cdot\, t)$. By [DL] we have $S(\tau) \subseteq h(\mu) \subseteq E_f$ where $\tau = \lim \mu^{\star n} \star \check{\mu}^{\star n}$. Hence

$$\lim_{n\to\infty} \| P_\tau P_\mu^n f - P_\mu^n f \|_2 = 0.$$

Applying once again the fact that $\{P_\mu^n f\}_{n\geq 1}$ is relatively compact we get

$$\lim_{n\to\infty} \| P_\mu^n P_{\check{\mu}}^n P_\mu^n f - P_\mu^n f \|_2 = 0.$$

Next we notice that

$$\lim_{n\to\infty} \| P_{\check{\mu}}^n P_\mu^n P_{\check{\mu}}^n P_\mu^n f - P_{\check{\mu}}^n P_\mu^n f \|_2 = 0$$

which gives $P_{\rho\star\rho} f = P_\rho f$ and consequently $\rho \star \rho = \rho$. This yields that ρ is the Haar measure of the subgroup $S(\rho)$. By Lemma 2 we have

$$S(\rho) = S(\check{\mu}^{\star n} \star \rho \star \mu^{\star n}) \supseteq S(\check{\mu}^{\star n} \star \mu^{\star n})$$

for any natural n. Hence,

$$S(\rho) \subseteq \overline{\bigcup_{n=1}^{\infty} S(\check{\mu}^{\star n} \star \mu^{\star n})} \subseteq S(\rho) \subseteq h(\mu).$$

To get the opposite inclusion $S(\rho) \supseteq h(\mu)$ we only must show that $S(\rho)$ is a normal subgroup of $G(\mu)$. We begin with the obvious inclusion $S(\mu) \subseteq gS(\rho)$ where $g \in S(\mu)$. It is also clear that if $g \in S(\mu)$ then the mappings $\Phi_g(x) = g^{-1}xg$ leave the set $S(\rho)$ invariant. Since mappings Φ_g are isometries with respect to an invariant metric on G we conclude $\Phi_g(S(\rho)) = S(\rho)$, so $S(\rho)$ is a normal subgroup of $G(\mu)$.

Repeating the above arguments we obtain $S(\tau) = S(\rho) = h(\mu)$. Applying (1) we conclude $S(\mu)h(\mu)S(\check{\mu}) = S(\check{\mu})h(\mu)S(\mu) = h(\mu)$ which finally gives $h(\mu) = G_{\mathrm{sym}}(\mu)$. ∎

Now we turn to semidirect products. Let $\mathbf{R}^d$, $d \geq 1$ be the d-dimensional vector space with a fixed norm $\|\cdot\|$, and K be a compact group of linear operators on $\mathbf{R}^d$. By $G_{K,d}$ we denote the semidirect product $K \otimes \mathbf{R}^d$. An element g of $G_{K,d}$ is usually represented as the pair (k_g, x_g). We recall that the group operation is defined by $g_1g_2 = (k_{g_1}, x_{g_1})(k_{g_2}, x_{g_2}) = (k_{g_1}k_{g_2}, x_{g_1} + k_{g_1}(x_{g_2}))$. Obviously, $G_{K,d}$ equipped with the product topology is a locally compact Polish group. It is well known (see [HR] page 211) that it is unimodular, and the product measure $\lambda = \lambda_K \otimes \lambda_d$ (λ_K is the normalized Haar measure on K, λ_d is the Lebesgue measure on $\mathbf{R}^d$) is a Haar measure on $G_{K,d}$. Applying results of [EG] we may easily show that the group $G_{SO(d),d}$ is amenable. In fact, let us consider the sequence of sets $U_n = SO(d) \times \{ x \in \mathbf{R}^d\colon\ \|x\| \leq n \}$. Clearly we have

$$\lim_{n\to\infty} \frac{\lambda(FU_n \triangle U_n)}{\lambda(U_n)} = 0$$

for any fixed compact set $F \subseteq G_{SO(d),d}$. The fact that in general groups $G_{K,d}$ have no invariant metrics is explained in the second part of the paper. Therefore [DL] and [B] do not answer the question if a strictly aperiodic and adapted $\mu \in P(G_{K,d})$ is scattered.

The second task of this note is to show that the characterization of concentrated measures provided by Theorem 1 in [B] is valid for some metric groups which do not possess invariant metrics. We will show that $\mu \in P(G_{\mathbf{T},2})$ is not scattered if and only if $h(\mu)$ is compact. In general it is not true that $h(\mu)$ is compact if μ is nonscattered. Examples of such μ's may be found by the reader in [BO] and [E].

Now we are in a position to formulate the main result of the paper.

THEOREM: *Let $\mu \in P(G_{K,d})$ be nonscattered and $G(\mu)$ be noncompact. Then there exists $x_0 \neq 0$ such that $k(x_0) = x_0$ whenever $(k, x) \in G_{\mathrm{sym}}(\mu)$ for some x.*

Proof: Let us notice that if $\zeta_n = (\tau_n, X_n)$ denotes the random walk on $G_{K,d}$, determined by the measure μ, then τ_n is the random walk on the group K generated by $\mu_K(\cdot) = \mu(\cdot \times \mathbf{R}^d)$.

Since μ is nonscattered, then by [C] for a fixed (small) $\varepsilon > 0$ there exist a radius $r > 0$ and a sequence $a_n \in G_{K,d}$ such that for all natural n the inequality $\mu^{\star n}(C_n) > 1 - \varepsilon$ holds, where $C_n = a_n(K \times B_r)$ and $B_r = \{x \in \mathbf{R}^d : \|x\| \le r\}$.

We may also assume that

$$\rho(K \times B_r) > 1 - \varepsilon \tag{2}$$

where ρ is the same as in Lemma 2. By (1) we get

$$\int \int \rho(g(K \times B_r)\tilde{g}^{-1})\, d\mu^{\star n}(g)\, d\mu^{\star n}(\tilde{g}) > 1 - \varepsilon$$

and consequently

$$\int_{C_n} \int_{C_n} \rho(g(K \times B_r)\tilde{g}^{-1})\, d\mu^{\star n}(g)\, d\mu^{\star n}(\tilde{g}) > 1 - 4\varepsilon.$$

Thus for any natural n there exist g_n, $\tilde{g}_n \in C_n$ so that

$$\begin{aligned} 1 - 4\varepsilon &< \rho(g_n(K \times B_r)\tilde{g}_n^{-1}) \\ &= \rho(\{(k_{g_n} k k_{\tilde{g}_n}^{-1}, x_{g_n} + k_{g_n}(x) - k_{g_n} k k_{\tilde{g}_n}^{-1}(x_{\tilde{g}_n})) : k \in K, \|x\| \le r\}) \\ &\le \rho(\{(k, x_{g_n} - k(x_{\tilde{g}_n}) + x) : k \in K, \|x\| \le cr\}) \end{aligned}$$

where $c = \sup\{\|k\| : k \in K\} < \infty$. Combining (2) and the above estimation we get

$$\rho(\{(k, x_{g_n} - k(x_{\tilde{g}_n}) + x) : k \in K, \|x\| \le cr\} \cap (K \times B_{cr})) > 1 - 5\varepsilon. \tag{3}$$

We notice that $\|x_{g_n} - x_{\tilde{g}_n}\| \le 2cr$ since g_n, $\tilde{g}_n \in C_n$. By Lemma 1 $\mu^{\star n}(K \times B_r) \to 0$, so $\|a_n\|$ tends to infinity. Hence we may assume that $\|x_{g_n}\|$, $\|x_{\tilde{g}_n}\| \ge R_n \to \infty$. Now, let us denote

$$K_n = \{k \in K : \|x_{g_n} - k(x_{\tilde{g}_n})\| \le 2cr\} \quad \text{and} \quad \rho_K(\cdot) = \rho(\cdot \times \mathbf{R}^d).$$

From (3) we get $\rho_K(K_n) \ge 1 - 5\varepsilon$. Let $n_j \to \infty$ be such that

$$\frac{x_{g_{n_j}}}{\|x_{g_{n_j}}\|} \to x_\varepsilon \qquad \left(\text{and} \quad \frac{x_{\tilde{g}_{n_j}}}{\|x_{\tilde{g}_{n_j}}\|} \to x_\varepsilon\right).$$

Clearly $\|x_\varepsilon\| = 1$ and $\|k(x_\varepsilon) - x_\varepsilon\| \leq \varepsilon$ for k taken from K_{n_j} (j large enough). As a result we get that for every $0 < \varepsilon < 1/5$ there exist a set $K_\varepsilon \subseteq K$ and $x_\varepsilon \in \mathbf{R}^d$ with $\|x_\varepsilon\| = 1$, such that $\rho_K(K_\varepsilon) > 1 - 5\varepsilon$ and $\|k(x_\varepsilon) - x_\varepsilon\| \leq \varepsilon$ for all $k \in K_\varepsilon$.

For some $\varepsilon_j \to 0$ we have $x_{\varepsilon_j} \to x_0$. Now let us consider

$$K_0 = \{k \in K: k(x_0) = x_0 \}.$$

The following estimation

$$\rho_K(K_0) \geq \rho_K(\{k \in K: \|k(x_{\varepsilon_j}) - x_{\varepsilon_j}\| < \varepsilon_j\}) > 1 - 5\varepsilon_j$$

holds if j is appropriate. We conclude $\rho_K(K_0) = 1$. Since K_0 is closed, it contains the set $S(\rho_K)$. By Lemma 3 we infer that $k(x_0) = x_0$ for any $k \in h(\mu_K) = G_{\mathrm{sym}}(\mu_K)$. To end the proof it is sufficient to notice that $G_{\mathrm{sym}}(\mu_K)$ consists of those element from K that for some $x \in \mathbf{R}^d$ one has $(k, x) \in G_{\mathrm{sym}}(\mu)$. ∎

COROLLARY 1: *Let μ be a probability measure on $G_{K,d}$. If $G(\mu)$ is noncompact and for every $x \in \mathbf{R}^d \setminus \{0\}$ there exist $(k_1, x_1), (k_2, x_2) \in S(\mu)$ with $k_1(x) \neq k_2(x)$, then μ is s.scattered.*

Since $G_{\mathrm{sym}}(\mu) = G$ is valid for all adapted and strictly aperiodic $\mu \in P(G)$, the next corollary provides a solution to the problem raised by Y. Derriennic and M. Lin in the end of the paper [DL]. Namely, we have:

COROLLARY 2: *Let μ be a probability measure on the group G of all rigid motions of the d-dimensional Euclidian space $\mathbf{R}^d$ (i.e. $G = G_{SO(d),d}$) such that $G_{\mathrm{sym}}(\mu) = G$. Then μ is s.scattered.*

Proof: Clearly $G(\mu)$ is noncompact. We notice that the group $SO(d)$ moves all nonzero vectors of $\mathbf{R}^d$ and the result follows by the equality $G_{\mathrm{sym}}(\mu_K) = SO(d)$. ∎

In this part of the paper we provide an example of a group whose left and right uniform structures are not equivalent (it has no invariant metric), but with the property that $\mu \in P(G)$ is nonscattered if and only if $h(\mu)$ is compact. By $\mathbf{T}$ we denote the circle group (i.e. $\mathbf{T} = \{z \in \mathbf{C}: |z| = 1\}$) with the ordinary group operation and topology. We will identify $\mathbf{R}^2$ with the complex plain $\mathbf{C}$.

Now let us consider the group $G_{\mathbf{T},2}(= G_{SO(2),2})$. We notice that there is no invariant metric on this group. In fact, let $g_n = (e^{i/n}, 0)$ and $y_n = (1, n)$. It is not difficult to check that $y_n g_n y_n^{-1} = (e^{i/n}, n(1 - \cos(1/n)) - n\sin(1/n)i)$ is not convergent to $(1, 0)$, the neutral element of $G_{\mathbf{T},2}$. Therefore by [HR],(8.18) there are no invariant metrics on this group.

PROPOSITION: *Let μ be a probability measure on $G_{\mathbf{T},2}$. Then μ is not scattered if and only if $h(\mu)$ is compact. Moreover, it holds only if either $G(\mu)$ is compact, or $\mu = \delta_{(1,z)}$ with $z \neq 0$ and then $G(\mu) = \mathbf{Z}$.*

Proof: Let us begin with the remark that if $h(\mu)$ is compact then without any restrictions on G the measure μ is nonscattered. In fact, for any natural n and $g \in S(\mu)$ we have $\mu^{\star n}(g^n h(\mu)) = 1$.

Now let us assume that μ is nonscattered and $G(\mu)$ is noncompact. By our Theorem there exists nonzero $x_0 \in \mathbf{C}$ so that $\alpha_1 x_0 = \alpha_2 x_0$ for all $(\alpha_1, z_1), (\alpha_2, z_2) \in S(\mu)$. Hence $\alpha_1 = \alpha_2$, which implies $\mu = \delta_\alpha \otimes \nu$ with ν a probability measure on $\mathbf{C}$. Let $X_1, X_2, \dots$ and $\tilde{X}_1, \tilde{X}_2, \dots$ be i.i.d. complex-valued random variables of distribution ν. Then the random walk ξ_n generated by μ has the representation $\xi_n = (\alpha^n, X_1 + \alpha X_2 + \cdots + \alpha^{n-1} X_n)$. Let

$$\tilde{\xi}_n = (\alpha^n, \tilde{X}_1 + \alpha \tilde{X}_2 + \cdots + \alpha^{n-1} \tilde{X}_n),$$

which is an independent copy of ξ_n. Since μ is assumed to be nonscattered, the sequence

$$\tilde{\zeta}_n^{-1} \zeta_n = (1, Y_n) = \Big(1, \overline{\alpha}^n \Big(\sum_{k=0}^{n-1} \alpha^k (X_{k+1} - \tilde{X}_{k+1}) \Big) \Big)$$

is stochastically convergent to $(1, Y)$ where Y is a complex-valued random variable (Lemma 2). If we denote by A_k and B_k the real and, respectively, the imaginary part of $X_k - \tilde{X}_k$, and represent α as $e^{i\tau}$, then elementary calculations lead us to

$$\Re(Y_n) = \sum_{k=0}^{n-1} (A_{k+1} \cos((n-k)\tau) + B_{k+1} \sin((n-k)\tau)),$$

and

$$\Im(Y_n) = \sum_{k=0}^{n-1} (A_{k+1} \sin((k-n)\tau) + B_{k+1} \cos((n-k)\tau)).$$

The characteristic functions $\Psi_{\Re(Y_n)}$ and $\Psi_{\Im(Y_n)}$ are pointwise convergent to the characteristic functions $\Psi_{\Re(Y)}$, $\Psi_{\Im(Y)}$, respectively. In particular, using independence we obtain the following formulae:

$$\Psi_{\Re(Y_n)}(t) = \prod_{k=0}^{n-1} \phi_{k,n}^{(\Re)}(t), \quad \Psi_{\Im(Y_n)}(t) = \prod_{k=0}^{n-1} \phi_{k,n}^{(\Im)}(t)$$

where $\phi_{k,n}^{(\Re)}$, $\phi_{k,n}^{(\Im)}$ stand for the characteristic functions of random variables $A_1 \cos((n-k)\tau) + B_1 \sin((n-k)\tau)$, and $A_1 \sin((k-n)\tau) + B_1 \cos((n-k)\tau)$. If $|\phi_{k,n}^{(\Re)}(t)| < 1 - \varepsilon$ ($|\phi_{k,n}^{(\Im)}(t)| < 1 - \varepsilon$) for some k, then the periodicity of the functions sin and cos implies that this takes place as many times as we wish, if only n is large enough. Thus $\Psi_{\Re(Y)}(t) = 0$ ($\Psi_{\Im(Y)}(t) = 0$) if for some k, n we have $|\phi_{k,n}^{(\Re)}(t)| < 1$ ($|\phi_{k,n}^{(\Im)}(t)| < 1$).

As a result, for any $t \in \mathbf{R}$ we have $|\Psi_{\Re(Y)}(t)|$ and $|\Psi_{\Im(Y)}(t)|$ are 0 or 1. The continuity of characteristic functions gives the identities $|\Psi_{\Re(Y)}| = |\Psi_{\Im(Y)}| \equiv 1$. We notice that may happen only if $\Re(Y)$ and $\Im(Y)$ are constant with probability one. Since Y and $-Y$ have the same distribution, we finally get $Y = 0$ with probability one. As a result we have the convergence

$$\lim_{n \to \infty} \mathrm{Prob}\Big(\Big|\overline{\alpha}^n \sum_{k=0}^{n-1} \alpha^k (X_{k+1} - \tilde{X}_{k+1})\Big| > \varepsilon\Big) = 0.$$

Therefore the series $\sum_{k=0}^{\infty} \alpha^k (X_{k+1} - \tilde{X}_{k+1})$ converges to 0 in probability which is only possible if X_1 is constant with probability one. This means that $\mu = \delta_{(\alpha,z)}$ for some $\alpha \in \mathbf{T}$ and $z \in \mathbf{C}$. It is not difficult to notice that holds for noncompact $G(\mu)$ only if $\alpha = 1$, and $z \neq 0$; then the group $G(\mu)$ is isomorphic to $\mathbf{Z}$. Clearly $h(\mu) = \{(1,0)\}$ in this case.

If $G(\mu)$ is compact then obviously $h(\mu)$ must be compact. The proof of the proposition is completed. ∎

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